Eigenvectors and eigenfunctionals of homogeneous order-preserving maps

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Abstract

This paper considers homogeneous order preserving continuous maps on the normal cone of an ordered normed vector space. It is shown that certain operators of that kind which are not necessarily compact themselves but have a compact power have a positive eigenvector that is associated with the cone spectral radius. We also derive conditions for the existence of homogeneous order preserving eigenfunctionals. Our results are illustrated in a model for spatially distributed two-sex populations.

Keywords: cone spectral radius, power compact, monotonically compact, order bounded, normal cone, beer barrel, two-sex models.

1 Introduction

For a linear bounded operator map B on a complex Banach space, the spectral radius of B is defined as

$$\mathbf{r}(B) = \sup\{|\lambda|; \lambda \in \sigma(B)\}, \tag{1.1}$$

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where $\sigma(B)$ is the spectrum of B,

$$\sigma(B) = \mathbb{C} \setminus \rho(B), \tag{1.2}$$

and $\rho(B)$ the resolvent set of B, i.e., the set of those $\lambda \in \mathbb{C}$ for which $\lambda - B$ has a bounded everywhere defined inverse. The following alternative formula holds,

$$\mathbf{r}(B) = \inf_{n \in \mathbb{N}} \|B^n\|^{1/n} = \lim_{n \to \infty} \|B^n\|^{1/n},$$

which is also meaningful in a real Banach space. If B is a compact linear map on a complex Banach space and $\mathbf{r}(B) > 0$, then there exists some $\lambda \in \sigma(B)$ and $v \in X$ such that $|\lambda| = \mathbf{r}(B)$ and $Bv = \lambda v \neq 0$. Such an λ is called an eigenvalue of B. This raises the question whether $\mathbf{r}(B)$ could be an eigenvalue itself. There is a positive answer, if B is a positive operator and satisfies some generalized compactness assumption.

1.1 Positivity

For models in the biological, social, or economic sciences, there is a natural interest in solutions that are positive in an appropriate sense.

A closed subset X_+ of a normed real vector space X is called a *wedge* if

(i) X_+ is convex,

(ii) $\alpha x \in X_+$ whenever $x \in X_+$ and $\alpha \in \mathbb{R}_+$.

A wedge is called a *cone* if

(iii) $X_+ \cap (-X_+) = \{0\}$.

Nonzero points in a cone or wedge are called *positive*.

A wedge is called *solid* if it contains interior points.

A wedge is called *reproducing* (also called *generating*) if

$$X = X_+ - X_+, \tag{1.3}$$

and *total* if X is the closure of $X_+ - X_+$.

A cone X_+ is called *normal*, if there exists some $\delta > 0$ such that

$$\|x + z\| \geq \delta \text{ whenever } x \in X_+, z \in X_+, \|x\| = 1 = \|z\|. \tag{1.4}$$

Equivalent conditions for a cone to be normal are given in Theorem 2.1. In function spaces, typical cones are formed by the nonnegative functions.

A map B on X with $B(X_+) \subseteq X_+$ is called a *positive map*.

If X_+ is a cone in X, we introduce a partial order on X by $x \le y$ if $y - x \in X_+$ for $x, y \in X$.

Definition 1.1. Let X and Z be ordered vector space with cones X_+ and Z_+ and $U \subseteq X$. A map $B : U \to Z$ is called *order preserving* (or monotone or increasing) if $Bx \le By$ whenever $x, y \in U$ and $x \le y$.

Positive linear maps are order-preserving. They have the remarkable property that their spectral radius is a spectral value [8] [36, App.2.2] if X is a Banach space and X_+ a normal reproducing cone.

The celebrated Krein-Rutman theorem [23], which generalizes parts of the Perron-Frobenius theorem to infinite dimensions, establishes that a compact positive linear map B with $\mathbf{r}(B) > 0$ on an ordered Banach space X with total cone X_+ has an eigenvector $v \in X_+$, $v \ne 0$, such that $Bv = \mathbf{r}(B)v$ and a positive bounded linear eigenfunctional $v^* : X \to \mathbb{R}$, $v \ne 0$, such that $x^* \circ B = \mathbf{r}(B)x^*$.

This theorem has been generalized into various directions by Bonsall [7] and Birkhoff [4], Nussbaum [28, 29], and Eveson and Nussbaum [14] (see these papers for additional references).

Can the Krein-Rutman theorem be extended to certain nonlinear maps on cones?

1.2 The space of bounded homogeneous maps

In the following, X, Y and Z are ordered normed vector spaces with cones X_+, Y_+ and Z_+ respectively,

Definition 1.2. $B : X_+ \to Y$ is called *(positively) homogeneous (of degree one)*, if $B(\alpha x) = \alpha Bx$ for all $\alpha \in \mathbb{R}_+$, $x \in X_+$.

Since we do not consider maps that are homogeneous in other ways, we will simply call them homogeneous maps. If follows from the definition that

$$B0 = 0.$$

Homogeneous maps are not Frechet differentiable at 0 unless $B(x + y) = Bx + By$ for all $x, y \in X_+$. For the following holds.

Proposition 1.3. *Let $B : X_+ \to Y$ be homogeneous. Then the directional derivatives of B exist at 0 in all directions of the cone and*

$$\partial B(0,x) = \lim_{t\to 0+} \frac{B(tx) - B(0)}{t} = B(x), \qquad x \in X_+.$$

There are good reasons to consider homogeneous maps. Here is a mathematical one.

Theorem 1.4. *Let $F : X_+ \to Y$ and $u \in X$. Assume that the directional derivatives of F at u exist in all directions of the cone. Then the map $B : X_+ \to X_+$, $B = \partial F(u,\cdot)$,*

$$B(x) = \partial F(u,x) = \lim_{t\to 0+} \frac{F(u+tx) - F(u)}{t}, \qquad x \in X_+,$$

is homogeneous.

Proof. Let $\alpha \in \mathbb{R}_+$. Obviously, if $\alpha = 0$, $B(\alpha x) = 0 = \alpha B(x)$. So we assume $\alpha \in (0,\infty)$. Then

$$\frac{F(u + t[\alpha x]) - F(u)}{t} = \alpha \frac{F(u + [t\alpha]x) - F(u)}{t\alpha}.$$

As $t \to 0$, also $\alpha t \to 0$ and so the directional derivative in direction αx exists and

$$\partial F(u, \alpha x) = \alpha F(u,x). \qquad \square$$

Another good reason are mathematical population models that take into account that, for many species, reproduction involves a mating process between two sexes. The map involved therein is not only homogeneous but also order preserving (Section 8). The spectral radius of a positive linear map has gained considerable notoriety because of its relation to the basic reproduction number of population models which have a highly dimensional structure but implicitly assume a one to one sex ratio [11, 13, 37, 40]. A spectral radius for homogeneous order-preserving maps should play a similar role as an extinction versus persistence threshold parameter for structured populations with two sexes.

For a homogeneous map $B : X_+ \to Y$, we define

$$\|B\|_+ = \sup\{\|Bx\|; x \in X_+, \|x\| \le 1\} \tag{1.5}$$

and call B *bounded* if this supremum is a real number. Since B is homogeneous,

$$\|Bx\| \le \|B\|_+ \, \|x\|, \qquad x \in X_+. \tag{1.6}$$

Let $H(X_+, Y)$ denote the set of bounded homogeneous maps $B : X_+ \to Y$ and $H(X_+, Y_+)$ denote the set of bounded homogeneous maps $B : X_+ \to Y_+$ and $\mathrm{HM}(X_+, Y_+)$ the set of those maps in $H(X_+, Y_+)$ that are also order-preserving.

$H(X_+, Y)$ is a real vector space and $\| \cdot \|_+$ is a norm on $H(X, Y_+)$; $H(X_+, Y_+)$ and $\mathrm{HM}(X_+, Y_+)$ are cones in $H(X_+, Y)$. We write $H(X_+) = H(X_+, X_+)$ and $\mathrm{HM}(X_+) = \mathrm{HM}(X_+, X_+)$.

It follows for $B \in H(X_+, Y_+)$ and $C \in H(Y_+, Z_+)$ that $CB \in H(X_+, Z_+)$ and

$$\|CB\|_+ \le \|C\|_+ \, \|B\|_+.$$

1.3 Cone and orbital spectral radius for bounded homogeneous maps

Let $B \in H(X_+)$ and define $\phi : \mathbb{Z}_+ \to \mathbb{R}$ by $\phi(n) = \ln \|B^n\|_+$. Then $\phi(m + n) \le \phi(m) + \phi(n)$ for all $m, n \in \mathbb{Z}_+$, and a well-known result implies the following formula for the *cone spectral radius*

$$\mathbf{r}_+(B) := \inf_{n \in \mathbb{N}} \|B^n\|_+^{1/n} = \lim_{n \to \infty} \|B^n\|_+^{1/n}. \tag{1.7}$$

Mallet-Paret and Nussbaum [25, 26] suggest an alternative definition of a spectral radius for homogeneous bounded maps $B : X_+ \to X_+$. First, define an asymptotic least upper bound for the geometric growth factor of B-orbits,

$$\gamma_B(x) := \limsup_{n \to \infty} \|B^n x\|^{1/n}, \qquad x \in X_+, \tag{1.8}$$

and then

$$\mathbf{r}_o(B) = \sup_{x \in X_+} \gamma_B(x). \tag{1.9}$$

The number $\mathbf{r}_+(B)$ has been called *partial spectral radius* by Bonsall [8], X_+ spectral radius by Schaefer [35, 36], and *cone spectral radius* by Nussbaum [29]. Mallet-Paret and Nussbaum [25, 26] call $\mathbf{r}_+(B)$ the *Bonsall cone spectral radius* and $\mathbf{r}_o(B)$ the cone spectral radius. For $x \in X_+$, the number $\gamma_B(x)$ has been called *local spectral radius* of B at x by Förster and Nagy [15].

We will follow Nussbaum's older terminology which shares the spirit with Schaefer's [35] term X_+ *spectral radius* and stick with *cone spectral radius* for $\mathbf{r}_+(B)$ and call $\mathbf{r}_o(B)$ the *orbital spectral radius* of B.

One readily checks that

$$\mathbf{r}_+(\alpha B) = \alpha \mathbf{r}_+(B), \quad \alpha \in \mathbb{R}_+, \qquad \mathbf{r}_+(B^m) = (\mathbf{r}_+(B))^m, \qquad m \in \mathbb{N}, \tag{1.10}$$

and that the same properties hold for $\mathbf{r}_o(B)$.

The cone spectral radius and the orbital spectral radius are meaningful if B is just positively homogeneous and bounded, but as in [25, 26] we will be mainly interested in the case that B is also order-preserving and continuous.

Though the two concepts coincide for many practical purposes, they are both useful.

Theorem 1.5. *Let X be an ordered normed vector with cone X_+ and $B : X_+ \to X_+$ be continuous, homogeneous and order preserving.*

Then $\mathbf{r}_+(B) \ge \mathbf{r}_o(B) \ge \gamma_B(x)$, $x \in X_+$.

Further $\mathbf{r}_o(B) = \mathbf{r}_+(B)$ if one of the following hold:

(i) X_+ is complete and normal.

(ii) A power of B is compact.

(iii) X_+ is normal and a power of B is uniformly order bounded.

For the concepts and the proof of (iii) see Section 3.4. The other two conditions for equality have been proved in [25], Theorem 2.2 and Theorem 2.3 (the overall assumption of [25] that X is a Banach space is not used in the proofs.)

For a homogeneous bounded map, the definitions of the cone and orbital spectral radius have somewhat lost the connection to the spectrum of the map which is difficult to generalize. But at least one would like to know whether $\mathbf{r}_+(B)$ or $\mathbf{r}_o(B)$ are eigenvalues of B, i.e., is there some $x \in X_+$, $x \ne 0$, such that $Bx = \mathbf{r}_+(B)x$ ($Bx = \mathbf{r}_o(B)x$).

Already the seminal work by Krein and Rutman [23] (see also [9, Thm.4.3]) establishes the existence of some $x \in X_+$, $x \ne 0$, and some $\lambda > 0$ such that $Bx = \lambda x$. B is assumed to be continuous, compact, and dominant with the latter meaning that there is some $c > 0$ and $u \in X_+$, $u \ne 0$, such that $Bu \ge cu$. The eigenvalue λ then turns out to satisfy $\lambda \ge c$ but no connection

is made between λ and some sort of spectral radius of B. The homogeneity of B can be dropped if B is assumed to be strictly positive in an appropriate sense and the cone is normal [34] (see also [9, Thm.4.2]).

Bohl [5] [6, III.2] also proves the existence of a positive eigenvector. Under his stronger assumptions it is clear that the eigenvalue λ is the cone and orbital spectral radius though he does not introduce these concepts either. Bohl assumes that the cone is solid, some power of B is compact and that B is order preserving in some strict sense (actually it is enough that B commutes with such a map). Bohl's proof is constructive as it provides the convergence of the (ratio) power method (von Mises procedure [41]) to the eigenvector x and the eigenvalue λ, and it also establishes convergence from below and above to the spectral radius by what are sometimes called Collatz-Wielandt numbers [10, 15, 42].

A little later, Nussbaum [30, 31, 32] and, more recently, Mallet-Paret and Nussbaum [25, 26] have found eigenvectors of homogeneous order preserving operators assuming different compactness and monotonicity properties than Krein/Rutman and Bohl and not requiring a dominance property of the map or the solidity of the cone. Mallet-Paret and Nussbaum [25, 26] also establish that the eigenvalue can be chosen as the orbital spectral radius.

Related to problems with beer barrel scent [27, 39], it appears that there are some questions left when B is not compact itself and does not satisfy monotonicity properties strong enough that B becomes a strict contraction with respect to Hilbert's projective metric [31, 32, 33]. Our approach is as old as the earliest literature in this direction [21, Sec.2.2] [34, Sec.3], namely to approximate B by operators B_n that have strong monotonicity properties. The crux of this approach is to what degree B_n inherits generalized compactness properties from B.

In the spirit of the Krein-Rutman theorem which involves the existence of eigenfunctionals as well, we also investigate whether there is a homogeneous, order preserving, bounded eigenfunctional $\phi : X_+ \to \mathbb{R}_+$ such that $\phi \circ B = \mathbf{r}_+(B)\phi \neq 0$.

2 More on cones

While the Krein-Rutman theorem and its generalizations to homogeneous maps do not require the cone to be normal, we found that we can make only little further progress if normality is not assumed.

2.1 Normal cones

The following result is well-known [21, Sec.1.2].

Theorem 2.1. *Let X be an ordered normed vector space with cone X_+. Then the following three properties are equivalent:*

(i) X_+ is normal: There exists some $\delta > 0$ such that $\|x+z\| \ge \delta$ whenever $x \in X_+$, $z \in X_+$ and $\|x\| = 1 = \|z\|$.

(ii) The norm is semi-monotonic: There exists some $M \ge 0$ such that $\|x\| \le M\|x+z\|$ for all $x, z \in X_+$.

(iii) There exists some $\tilde{M} \ge 0$ such that $\|x\| \le \tilde{M}\|y\|$ whenever $x \in X$, $y \in X_+$, and $-y \le x \le y$.

Remark 2.2. If X_+ were just a wedge, property (iii) would be rewritten as

> There exists some $\tilde{M} \ge 0$ such that $\|x\| \le \tilde{M}\|y\|$ whenever $x \in X$, $y \in X_+$, and $y + x \in X_+$, $y - x \in X_+$.

Notice that this property implies that X_+ is cone: If $x \in X_+$ and $-x \in X_+$, then $0 + x \in X_+$ and $0 - x \in X_+$ and (iii) implies $\|x\| \le \tilde{M}\|0\| = 0$.

Proposition 2.3 (cf. [22, (4.2)])**.** *Let X be an ordered normed vector space with cone X_+. We define $\psi : X \to \mathbb{R}_+$ by*

$$\psi(x) = \inf\{\|y\|; y \in X, y \ge x\}, \qquad x \in X. \tag{2.1}$$

Then ψ is homogenous, order preserving, and subadditive ($\psi(x+y) \le \psi(x) + \psi(y)$, $x, y \in X$),

$$\begin{aligned} |\psi(x) - \psi(y)| \le \|x - y\|, &\qquad x, y \in X, \\ \psi(x) = 0, &\qquad x \in -X_+. \end{aligned}$$

Moreover, ψ is strictly positive: $\psi(x) > 0$ for all $x \in X_+$, $x \ne 0$.

If X_+ is normal, there exists some $\delta > 0$ such that $\delta\|x\| \le \psi(x) \le \|x\|$ for all $x \in X_+$.

Proof. The functional ψ inherits positive homogeneity from the norm. That ψ is order-preserving is immediate from the definition. For all $x \in X$, $x \leq x$ and so $\|x\| \geq \psi(x)$.

To show the subadditivity, let $u, v, x, y \in X$ and $u \geq x$ and $v \geq y$. Then $u + v \geq x + y$ and $\psi(x + y) \leq \|u + v\| \leq \|u\| + \|v\|$. Fix u for a moment. Then $\psi(x + y) - \|u\| \leq \|v\|$ for all $v \geq y$. Thus $\psi(x + y) - \|u\| \leq \psi(y)$. Hence $\psi(x + y) - \psi(y) \leq \|u\|$ for all $u \geq x$ and $\psi(x + y) - \psi(y) \leq \psi(x)$.

Since ψ is subadditive,

$$|\psi(x) - \psi(y)| \leq \psi(x - y) \leq \|x - y\|.$$

To show that ψ is strictly positive, let $x \in X_+$ and $\psi(x) = 0$. By definition, there exists a sequence (y_n) in X with $\|y_n\| \to 0$ and $y_n \geq x$ for all $n \in \mathbb{N}$. Then $y_n - x \in X_+$. Since X_+ is closed, $-x = \lim_{n\to\infty}(y_n - x) \in X_+$. Since $x \in X_+$, $x = 0$.

Assume that X_+ is normal. Then there exists some $c > 0$ such that $\|y\| \leq c\|x\|$ whenever $x, y \in X_+$ and $y \leq x$. Hence $\|x\| \leq c\psi(x)$ for all $x \in X_+$. Set $\delta = 1/c$. □

Theorem 2.4 (cf. [22, Thm.4.4])**.** *Let X be an ordered normed vector space with normal cone X_+. Define*

$$\|x\|^\diamond = \max\{\psi(x), \psi(-x)\}, \qquad x \in X,$$

with ψ from Proposition 2.3. Then $\|\cdot\|^\diamond$ is a norm on X that is equivalent to the original norm (actually $\|x\|^\diamond \leq \|x\|$ for all $x \in X_+$). Further $\|\cdot\|^\diamond$ is order preserving on X_+: $\|x\|^\diamond \leq \|y\|^\diamond$ for all $x, y \in X_+$ with $x \leq y$. Finally, for all $x, y, z \in X$ with $x \leq y \leq z$,

$$\|y\|^\diamond \leq \max\{\|x\|^\diamond, \|z\|^\diamond\}.$$

Proof. It is easy to see from the properties of ψ that $\|\alpha x\|^\diamond = |\alpha|\|x\|^\diamond$ for all $\alpha \in \mathbb{R}$, $x \in X$, and that $\|\cdot\|^\diamond$ is subadditive. Now $\psi(x) \leq \|x\|$ and $\psi(-x) \leq \|-x\| = \|x\|$ and so $\|x\|^\diamond \leq \|x\|$.

That $\|\cdot\|$ and $\|\cdot\|^\diamond$ are equivalent norms is shown in [22, Thm.4.4].

To prove the last statement, let $x, y, z \in X$ and $x \leq y \leq z$. Then $y \leq z$ and $-y \leq -x$. Since ψ is order preserving,

$$\psi(y) \leq \psi(z) \leq \|z\|^\diamond, \qquad \psi(-y) \leq \psi(-x) \leq \|x\|^\diamond.$$

This implies the assertion. □

Corollary 2.5. *Let X be an ordered normed vector space with normal cone X_+. Then there exists some $c \geq 0$ such that $\|y\| \leq c \max\{\|x\|, \|z\|\}$ for all $x, y, z \in X_+$ with $x \leq y \leq z$.*

Proof. Let $\| \cdot \|^\diamond$ be the equivalent norm from Theorem 2.4 and $c \geq 0$ such that $\|x\|^\diamond \leq \|x\| \leq c\|x\|^\diamond$ for all $x \in X$. Let $x \leq y \leq z$. Then

$$\|y\| \leq c\|y\|^\diamond \leq c \max\{\|x\|^\diamond, \|z\|^\diamond\} \leq c \max\{\|x\|, \|z\|\}. \qquad \square$$

Corollary 2.6 (Squeezing theorem [22, Thm.4.3])**.** *Let X be an ordered normed vector space with a normal cone X_+. Let $y \in X$ and (x_n), (y_n), (z_n) be sequences in X with $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $x_n \to y$ and $z_n \to y$. Then $y_n \to y$.*

Proof. Notice that $x_n - y \leq y_n - y \leq z_n - y$. By Corollary 2.5, with some $c \geq 0$ that does not depend on n,

$$\|y_n - y\| \leq c \max\{\|z_n - y\|, \|x_n - y\|\} \to 0. \qquad \square$$

2.2 (Fully) regular cone

Definition 2.7. Let X be an ordered normed vector space with cone X_+.

X_+ is called *regular* if any decreasing sequence in X_+ converges.

X_+ is called *fully regular* if any increasing bounded sequence in X_+ converges.

The norm of X is called *additive* on X_+ if $\|x + z\| = \|x\| + \|z\|$ for all $x, z \in X_+$.

Theorem 2.8. *Let X be an ordered normed vector space with cone X_+.*

(a) If X_+ is complete and regular, then X_+ is normal.

(b) If X_+ is complete and fully regular, then X_+ is normal.

(c) If X_+ is normal and fully regular, then X_+ is regular.

(d) If X_+ is complete and fully regular, then X_+ is regular.

Proof. Notice that the proofs in [21, 1.5.2] and [21, 1.5.3] only need completeness of X_+ and not of X. $\square$

Theorem 2.9. *Let X be an ordered normed vector space with cone X_+. If X_+ is complete with additive norm, then X_+ is fully regular.*

Proof. Let (x_n) be an increasing sequence in X_+ such that there is some $c > 0$ such that $\|x_n\| \leq c$ for all $n \in N$. Define $y_n = x_{n+1} - x_n$. Then $y_n \in X_+$ and $\sum_{k=j}^m y_n = x_{m+1} - x_j$. Since the norm is additive on X_+,

$$\sum_{k=1}^m \|y_n\| = \Big\| \sum_{k=1}^m y_n \Big\| = \|x_{m+1} - x_1\| \leq 2c, \qquad m \in \mathbb{N}.$$

This implies that (x_n) is a Cauchy sequence in the complete cone X_+ and thus converges. □

The standard cones of the Banach spaces $L^p(\Omega)$, $1 \leq p < \infty$, are regular and completely regular, while the cones of $BC(\Omega)$, the Banach space of bounded continuous functions, and of $L^\infty(\Omega)$ are neither regular nor completely regular though normal.

2.3 The space of certain order-bounded elements and some functionals

Definition 2.10. Let $x \in X$ and $u \in X_+$. Then x is called *u-bounded* if there exists some $c > 0$ such that $-cu \leq x \leq cu$. If x is u-bounded, we define

$$\|x\|_u = \inf\{c > 0; -cu \leq x \leq cu\}. \tag{2.2}$$

The set of u-bounded elements in X is denoted by X_u. If $x, u \in X_+$ and x is not u-bounded, we define

$$\|x\|_u = \infty.$$

Two elements x and u in X_+ are called *comparable* if x is u-bounded and u is x-bounded.

If X is a space of real-valued functions on a set Ω,

$$\|x\|_u = \sup\Big\{\frac{|x(\xi)|}{u(\xi)}; \xi \in \Omega, u(\xi) > 0\Big\}.$$

Since the cone X_+ is closed,

$$-\|x\|_u u \leq x \leq \|x\|_u u, \qquad x \in X_u. \tag{2.3}$$

X_u is a linear subspace of X, $\|\cdot\|_u$ is a norm on X_u, and X_u, under this norm, is an ordered normed vector space with cone $X_+ \cap X_u$ which is normal, reproducing, and has nonempty interior.

If X_+ is normal, by Theorem 2.1, there exists some $M \geq 0$ such that

$$\|x\| \leq M\|x\|_u\|u\|, \qquad x \in X_u. \tag{2.4}$$

If X_+ is a normal and complete cone of X, then $X_+ \cap X_u$ is a complete subset of X_u with the metric induced by the norm $\|\cdot\|_u$. For more information see [21, 1.3] [6, I.4], [22, 1.4].

For $u \in X_+$, one can also consider the functionals

$$\left.\begin{array}{rl} (x/u)^\diamond = & \inf\{\alpha \in \mathbb{R}; x \leq \alpha u\} \\ (x/u)_\diamond = & \sup\{\beta \in \mathbb{R}; \beta u \leq x\} \end{array}\right\} x \in X,$$

with the convention that $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = -\infty$. If X is a space of real-valued functions on a set Ω,

$$\left.\begin{array}{rl} (x/u)^\diamond = & \sup\left\{\frac{x(\xi)}{u(\xi)}; \xi \in \Omega, u(\xi) > 0\right\} \\ (x/u)_\diamond = & \inf\left\{\frac{x(\xi)}{u(\xi)}; \xi \in \Omega, u(\xi) > 0\right\} \end{array}\right\} x \in X.$$

Many other symbols have been used for these two functionals in the literature; see Thompson [38] and Bauer [3] for some early occurrences. For $x \in X_+$, $\|x\|_u = (x/u)^\diamond$. Since we will use this functional for $x \in X_+$ only, we will stick with the notation $\|x\|_u$. Again for $x \in X_+$, $(x/u)_\diamond$ is a nonnegative real number, and we will use the leaner notation

$$[x]_u = \sup\{\beta \geq 0; \beta u \leq x\}, \qquad x, u \in X_+. \tag{2.5}$$

Since the cone X_+ is closed,

$$x \geq [x]_u u, \qquad x, y \in X_+. \tag{2.6}$$

Further $[x]_u$ is the largest number for which this inequality holds. The functional $[\cdot]_u : X_+ \to \mathbb{R}_+$ is homogeneous and concave.

3 More on homogeneous order-preserving maps and their spectral radii

If the cones have appropriate properties, homogeneous order-preserving maps are automatically bounded.

Theorem 3.1. *Let X and Y be ordered normed vector spaces with cones X_+ and Y_+. Let X_+ be a complete or fully regular cone, Y_+ a normal cone, and $B : X_+ \to Y_+$ be homogeneous and order-preserving. Then B is continuous at 0 and in particular bounded.*

The proof is adapted from [22, Thm.2.1].

Proof. Assume that B is not continuous at 0. Then there exists some $\epsilon > 0$ and a sequence $(x_n) \in X_+$ such that $\|x_n\| \leq 2^{-2n}$ and $\|Bx_n\| \geq \epsilon$ for all $n \in \mathbb{N}$. Since X_+ is complete or fully regular, the series $\sum_{n=1}^{\infty} 2^n x_n =: w$ converges in X_+. Then $0 \leq 2^n x_n \leq w$ for all $n \in \mathbb{N}$. Since B is positively homogeneous and order-preserving, $0 \leq Bx_n \leq 2^{-n} Bw$. Since Y_+ is normal, $Bx_n \to 0$. This contradiction shows that B is continuous at 0 and easily shown to be bounded. $\square$

Corollary 3.2. *Let X be an ordered normed vector space with a normal cone X_+ that is complete or fully regular. Let $B : X_+ \to X_+$ be homogeneous and order-preserving. Then B is continuous at 0 and in particular bounded.*

3.1 Monotonicity of the cone and orbital spectral radii

If the cone X_+ is normal, the cone and orbital spectral radius are increasing functions of the homogeneous bounded order-preserving maps (cf. [1, L.6.5])

Theorem 3.3. *Let X be an ordered normed vector space with a normal cone X_+ and $A, B : X_+ \to X_+$ be bounded and homogeneous. Assume that $Ax \leq Bx$ for all $x \in X_+$ and that A or B are order preserving. Then $\mathbf{r}_+(A) \leq \mathbf{r}_+(B)$ and $\mathbf{r}_o(A) \leq \mathbf{r}_o(B)$.*

Proof. We claim that $A^n x \leq B^n x$ for all $x \in X_+$ and all $n \in \mathbb{N}$. For $n = 1$, this holds by assumption. Now let $n \in \mathbb{N}$ and assume the statement holds for n. If A is order-preserving, then, for all $x \in X_+$, since $B^n x \in X_+$,

$$A^{n+1}x = AA^n x \leq AB^n x \leq BB^n x = B^{n+1}x.$$

If B is order-preserving, then, for all $x \in X_+$, since $A^n x \in X_+$,

$$A^{n+1}x = AA^n x \leq BA^n x \leq BB^n x = B^{n+1}x.$$

Since X_+ is normal, there exists some $c > 0$ such that $\|A^n x\| \leq c\|B^n x\|$ for all $x \in X_+$, $n \in \mathbb{N}$. Thus, $\|A^n\|_+ \leq c\|B^n\|_+$ for all $n \in \mathbb{N}$ and $\mathbf{r}_+(A) \leq \mathbf{r}_+(B)$. Further $\gamma_A(x) \leq \gamma_B(x)$ and so $\mathbf{r}_o(A) \leq \mathbf{r}_o(B)$. $\square$

For a bounded positive linear operator on an ordered Banach space, the spectral radius and cone spectral radius coincide provided that the cone is reproducing [26, Thm.2.14]. This is not true if the cone is only total [8, Sec.2,8].

Proposition 3.4. *Let X be an ordered Banach space and X_+ be reproducing. Then any linear positive operator $B : X \to Y$ that is bounded on X_+ is bounded. Further there exists some $c \geq 1$ such that, for all bounded linear positive maps B on X,*

$$\|B\|_+ \leq \|B\| \leq c\|B\|_+.$$

If $X = Y$, $\mathbf{r}_+(B) = \mathbf{r}(B)$.

3.2 A uniform boundedness principle

Theorem 3.5. *Let X_+ be a normal complete cone of a normed vector space. Let $\{B_j; j \in J\}$ be an indexed family of continuous homogeneous order preserving maps $B_j : X_+ \to X_+$. Assume that, for each $x \in X_+$, $\{B_j(x); j \in J\}$ is a bounded subset of X_+. Then $\{\|B_j\|_+; j \in J\}$ is a bounded subset of $\mathbb{R}$.*

Proof. By assumption,

$$X_+ = \bigcup_{n\in\mathbb{N}} M_n \qquad M_n = \bigcap_{j\in J} \tilde{M}_{n,j}, \qquad \tilde{M}_{n,j} = \{x \in X_+; \|B_j(x)\| \leq n\}.$$

Since each B_j is continuous, $\tilde{M}_{n,j}$ is a closed subset of X_+ for all $n, j \in \mathbb{N}$. Then M_n is a closed subset of X_+ as an intersection of closed sets. Since X_+ is complete by assumption, by the Baire category theorem, there exists some $n \in \mathbb{N}$ such that M_n has nonempty interior. So there exists some $z \in X_+$ and $\epsilon > 0$ such that

$$z + \epsilon y \in M_n \qquad \text{whenever} \quad y \in X, z + \epsilon y \in X_+, \|y\| \leq 1.$$

Since $z + \epsilon y \in X_+$ if $y \in X_+$,

$$\|B_j(z + \epsilon y)\| \leq n, \qquad y \in X_+, \|y\| \leq 1, j \in J.$$

Let $y \in X_+$, $\|y\| \leq 1$, $j \in J$. Since B_j is homogeneous and order preserving,

$$\epsilon B_j(y) = B_j(\epsilon y) \leq B_j(z + \epsilon y).$$

Since X_+ is normal, there exists some $c \ge 0$ (independent of y and j) such that

$$\|\epsilon B_j(y)\| \le c\|B_j(z+\epsilon y)\| \le cn.$$

Thus

$$\|B_j(y)\| \le \frac{cn}{\epsilon}, \qquad y \in X_+, \|y\| \le 1, j \in J.$$

By definition of $\|\cdot\|_+$,

$$\|B_j\|_+ \le \frac{cn}{\epsilon}, \qquad j \in J.$$

□

3.3 A left resolvent

There is one remnant from the usual relations between the spectral radius and the spectrum of a linear operator that also holds in the homogeneous case, namely that real numbers larger than the spectral radius are in the resolvent set. However, in the homogeneous case, there only exists a left resolvent.

Let X be a normed vector space with a cone X_+ which is complete or fully regular. Let $B : X_+ \to X_+$ be homogeneous, continuous and order preserving.

For $\lambda > \mathbf{r}_+(B)$, we introduce $R_\lambda : X_+ \to X_+$,

$$R_\lambda(x) = \sum_{n=0}^{\infty} \lambda^{-n-1} B^n(x), \qquad x \in X_+. \tag{3.1}$$

The convergence of the series follows from the completeness or the full regularity of the cone. Then R_λ acts as a left resolvent,

$$\begin{aligned} R_\lambda(Bx) =& \sum_{n=0}^{\infty} \lambda^{-(n+1)} B^{n+1}x = \sum_{n=1}^{\infty} \lambda^{-n} B^n x \\ =& \lambda R_\lambda(x) - x, \qquad x \in X_+. \end{aligned} \tag{3.2}$$

It follows from the Weierstraß majorant test that the convergence of the series is uniform for x in bounded subsets of X_+. With this in mind, the following is easily shown.

Lemma 3.6. *For $\lambda > \mathbf{r}_+(B)$, R_λ is defined, homogeneous, continuous, and order preserving.*

The next result suggests that $\mathbf{r}_+(B)$ is not an element of the resolvent set of B if B is also superadditive.

Theorem 3.7. *Let X be an ordered normed vector space with a normal cone X_+ that is complete or fully regular. Let B be homogeneous, order-preserving, bounded and $B(x+y) \geq B(x) + B(y)$ for all $x, y \in X_+$. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then $\|R_\lambda\|_+ \to \infty$ as $\lambda \to \mathbf{r}+$.*

Since B is homogeneous, the superadditivity assumption for B is equivalent to B being concave: $B((1-\alpha)x + \alpha y) \geq (1-\alpha)B(x) + \alpha B(y)$ for $\alpha \in (0,1)$. The proof is adapted from [8] and allows B to be concave rather than additive.

Proof. Since X_+ is normal, $\|R_\lambda\|_+$ is a decreasing function of $\lambda > \mathbf{r} := \mathbf{r}_+(B)$. R_λ inherits superadditivity from B,

$$R_\lambda(x+y) \geq R_\lambda(x) + R_\lambda(y), \qquad x, y \in X_+. \tag{3.3}$$

Suppose that the assertion is false. Then there exists some $M \geq 0$ such that $\|R_\lambda\|_+ \leq M$ for all $\lambda > \mathbf{r}$. Let $0 < \mu < \mathbf{r} < \lambda$ and $(\lambda - \mu)M < 1$. Since X_+ is complete or fully regular,

$$E_\mu(x) = \sum_{k=1}^\infty (\lambda - \mu)^{k-1} R_\lambda^k x$$

converges for each $x \in X_+$. By (3.2),

$$E_\mu(\lambda x) = \sum_{k=1}^\infty (\lambda - \mu)^{k-1} R_\lambda^k (\lambda x) = \sum_{k=1}^\infty (\lambda - \mu)^{k-1} R_\lambda^{k-1} (R_\lambda(Bx) + x).$$

Since R_λ is superadditive,

$$\begin{aligned} E_\mu(\lambda x) \geq & \sum_{k=1}^\infty (\lambda - \mu)^{k-1} R_\lambda^k (Bx) + \sum_{k=1}^\infty (\lambda - \mu)^{k-1} R_\lambda^{k-1}(x) \\ = & E_\mu(Bx) + x + (\lambda - \mu) E_\mu(x). \end{aligned}$$

Since E_μ is homogeneous,

$$E_\mu x \geq \frac{1}{\mu} x + \frac{1}{\mu} E_\mu B x, \qquad x \in X_+.$$

By iteration and induction,

$$E_\mu x \geq \sum_{k=0}^{n} \frac{1}{\mu^{k+1}} B^k x + \frac{1}{\mu^{n+1}} E_\mu B^{n+1} x, \qquad x \in X_+, n \in \mathbb{N}.$$

This implies that

$$E_\mu(x) \geq \frac{1}{\mu^{k+1}} B^k x, \qquad x \in X_+, n \in \mathbb{N}.$$

Since X_+ is normal, there exists some $\tilde{M} > 0$ (which depends on μ) such that

$$\|B^k x\| \leq \tilde{M} \mu^k \|x\|, \qquad k \in \mathbb{N}, x \in X_+.$$

Thus

$$\|B^k\|_+ \leq \tilde{M} \mu^k, \qquad k \in \mathbb{N}.$$

Since $\mu < \mathbf{r} = \mathbf{r}_+(B)$, this is a contradiction. □

Corollary 3.8. *Let X be an ordered normed vector space with a complete normal cone X_+. Let B be homogeneous, order-preserving, continuous and $B(x+y) \geq B(x) + B(y)$ for all $x, y \in X_+$. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then there exist some $x \in X_+$ and $x^* \in X_+^*$ such that $x^*(R_\lambda x) \to \infty$ as $\lambda \to \mathbf{r}+$.*

Proof. By the uniform boundedness principle in Theorem 3.5, there exists some $x \in X_+$ such that $\{\|R_\lambda x\|; \lambda > \mathbf{r}_+(B)\}$ is unbounded. By the usual uniform boundedness principle, applied to the Banach space X^*, there exists some $x^* \in X^*$ such that $\{|x^*(R_\lambda x)|; \lambda > \mathbf{r}_+(B)\}$ is unbounded. Since X_+ is normal, X_+^* is reproducing and the unboundedness holds for some $x^* \in X_+^*$. Since $x^*(R_\lambda x)$ is a decreasing function of λ, $x^*(R_\lambda x) \to \infty$ as $\lambda \to \mathbf{r}_+(B)$. □

3.4 Pointwise root power approximations of the spectral radius

In [2], it is suggested for a linear positive operator B to approximate $\mathbf{r}(B)$ by $\|B^n x\|^{1/n}$, $x \in X_+, x \neq 0$, for a special class of operators, but no general assumptions are specified within the paper that make this procedure work. Whenever this is possible for a homogeneous bounded map B on X_+, $\mathbf{r}_+(B) = \gamma_B(x) = \mathbf{r}_o(B$.

We can assume without loss of generality that $\mathbf{r}_+(B) > 0$, otherwise this equality trivially holds for all $x \in X_+$.

Definition 3.9. Let $B : X_+ \to X_+$, $u \in X_+$. B is called *pointwise u-bounded* if for any $x \in X_+$ there exists some $n \in \mathbb{N}$ and $\gamma > 0$ such that $B^n x \leq \gamma u$.

B is called *uniformly u-bounded* if there exists some $c > 0$ such that $Bx \leq c\|x\|u$ for all $x \in X_+$. The element u is called an *order bound* of B.

B is called *uniformly order bounded* if it is uniformly u-bounded for some $u \in X_+$. B is called *pointwise order bounded* if it is pointwise u-bounded for some $u \in X_+$.

This terminology has been adapted from various works by Krasnosel'skii [21, Sec.2.1.1] and coworkers [22, Sec.9.4] though it has been modified.

If $B : X_+ \to X_+$ is bounded and X_+ is solid, then B is uniformly u-bounded for every interior point u of X_+.

Proposition 3.10. *Let X be a normed vector space with complete cone X_+ and $B : X_+ \to X_+$ be continuous, order preserving and homogeneous. Let $u \in X_+$ and B be pointwise u-bounded. Then some power of B is uniformly u-bounded.*

Proof. Define

$$M_{n,k} = \{x \in X_+; B^n x \leq ku\}, \qquad n, k \in \mathbb{N}.$$

Since B is continuous and X_+ is closed, each set $M_{n,k}$ is closed. By assumption, $X_+ = \bigcup_{k,n\in\mathbb{N}} M_{n,k}$. Since X_+ is a complete metric space, by the Baire category theorem, there exists some $n, k \in \mathbb{N}$ such that $M_{n,k}$ contains a relatively open subset of X_+: There exists some $y \in X_+$ and $\epsilon > 0$ such that $y + \epsilon z \in M_{n,k}$ whenever $z \in X$, $\|z\| \leq 1$, and $y + \epsilon z \in X_+$. Now let $z \in X_+$ and $\|z\| \leq 1$. Since B is order preserving and $y + \epsilon z \in X_+$, $B^n(\epsilon z) \leq B^n(y + \epsilon z) \leq ku$. Since B is homogeneous, for all $x \in X_+$, $x \neq 0$,

$$B^n x = \frac{\|x\|}{\epsilon} B^n\Big(\frac{\epsilon}{\|x\|}x\Big) \leq \frac{k}{\epsilon}\|x\|u. \qquad \square$$

Theorem 3.11. *Let X be a normed vector space with a normal cone X_+ and $B : X_+ \to X_+$ be a homogeneous, bounded, order-preserving operator. Let $u \in X_+$.*

(a) Assume that B^m is uniformly u-bounded for some $m \in \mathbb{Z}_+$. Then

$$\mathbf{r}_+(B) = \lim_{n\to\infty} \|B^n u\|^{1/n}.$$

In particular, $\mathbf{r}_+(B) = \gamma_B(u) = \mathbf{r}_o(B)$.

(b) *Assume that* B *is pointwise* u*-bounded. Then*

$$\mathbf{r}_o(B) = \lim_{n\to\infty} \|B^n u\|^{1/n} = \gamma_B(u).$$

The case $m = 0$ in part (a) (the identity map is uniformly u-bounded) is proved in [25, Thm.2.2] under the additional assumption that B is continuous.

Proof. (a) Since B^m is uniformly u-bounded, there exists some $\gamma > 0$ such that

$$B^m x \le \gamma \|x\| u, \qquad x \in X_+.$$

Since B is homogeneous and order-preserving, for all $n \in \mathbb{N}$,

$$0 \le B^{m+n} x \le \gamma \|x\| B^n u, \qquad x \in X_+.$$

Since X_+ is normal, there exists some $\tilde\gamma > 0$ such that $\|B^{m+n}x\| \le \tilde\gamma \|x\| \, \|B^n u\|$ for all $n \in \mathbb{N}$, $x \in X$. So $\|B^{m+n}\|_+^{1/n} \le \tilde\gamma^{1/n} \|B^n u\|^{1/n}$. We can assume that $\mathbf{r}_+(B) > 0$. Then $\liminf_{t\to\infty} \|B^n u\|^{1/n} \ge \liminf_{n\to\infty} \|B^n\|_+^{1/n} = \mathbf{r}_+(B)$.

The proof of part (b) is similar. □

We also obtain upper estimates of the cone spectral radius.

Corollary 3.12. *Let* X *be a normed vector space with a normal cone* X_+ *and* $B : X_+ \to X_+$ *be a homogeneous, bounded, order-preserving operator. Let* $u \in X_+$*,* $\alpha \in \mathbb{R}_+$ *and* $k \in \mathbb{N}$ *such that* $B^k u \le \alpha^k u$*.*

(a) *If* B^m *is uniformly* u*-bounded for some* $m \in \mathbb{N}$*, then* $\mathbf{r}_+(B) \le \alpha$*.*

(b) *If* B *is pointwise* u*-bounded, then* $\mathbf{r}_o(B) \le \alpha$*.*

Proof. Since B is order-preserving and homogeneous, $B^{k\ell} u \le \alpha^{k\ell} u$ for all $\ell \in \mathbb{N}$. Since X_+ is normal, there exists some $c > 0$ such that $\|B^{k\ell}u\| \le c\alpha^{k\ell}\|u\|$ for all $\ell \in \mathbb{N}$. Now apply Theorem 3.11. □

Lower estimates of the cone spectral radius can be obtained under less assumptions. In particular, the cone does not need to be normal. Notice that the proof of [24, L.2.2] works if B is just bounded and order preserving but not necessarily continuous.

Theorem 3.13. *Let* X *be a normed vector space with cone* X_+*. Let* $B : X_+ \to X_+$ *be homogeneous, bounded, and order preserving. Further let* $x \in X_+$*,* $m \in \mathbb{N}$*, and* $\alpha \ge 0$ *such that* $B^m x \ge \alpha^m x$*. Then* $\mathbf{r}_+(B) \ge \mathbf{r}_o(B) \ge \alpha$*.*

4 Eigenvectors for compact maps

Recall the large-time bound of the geometric growth factor for initial value $u \in X_+$,

$$\gamma_B(u) := \limsup_{n\to\infty} \|B^n u\|^{1/n}. \tag{4.1}$$

Proposition 4.1. *Let X be an ordered normed vector space with normal cone X_+ and $B : X_+ \to X_+$ be homogeneous, order preserving, and bounded. Let $K : X_+ \to X_+$ be continuous and compact and $K(x) \geq Bx$ for all $x \in X_+$, $\|x\| = 1$. Finally let $u \in X_+$, $u \neq 0$.*

Then there exists some $x \in X_+$, $\|x\| = 1$, and some $\lambda \geq \gamma_B(u)$ such that $Kx + u = \lambda x$.

The idea of perturbing K and using one of the classical fixed point theorems seems as old as the theory of positive operators [21, Sec.2.2] [34, Sec.3].

Proof. Define

$$K_u(x) = K(x) + u, \qquad x \in X_+.$$

Then $K_u(x) \geq u$ and $\{\|K_u(x)\|; x \in X_+\}$ is bounded away from 0 because X_+ is closed. We can define

$$\tilde{K}(x) = \frac{K_u(x)}{\|K_u(x)\|}, \qquad x \in X_+.$$

Then $\tilde{K}$ is continuous and maps the set $C = X_+ \cap \bar{U}_1$, $\bar{U}_1 = \{x \in X; \|x\| \leq 1\}$, into itself. C is a closed convex subset of X and $\tilde{K}(C)$ has compact closure. By Tychonov's fixed point theorem [12, Thm.10.1], $\tilde{K}$ has a fixed point $x \in X_+$, $\|x\| = 1$,

$$K(x) + u = \lambda x, \qquad \lambda = \|K(x) + u\| > 0.$$

Since $K(x) \in X_+$, $u \leq \lambda x$. By assumption, $Bx \leq \lambda x$. Since B is order preserving and homogeneous, $B^n x \leq \lambda^n x$ for all $n \in \mathbb{N}$. Hence $B^n u \leq \lambda^n \lambda x$. Since X_+ is normal and $\|x\| = 1$, there exists some $\delta_x > 0$ such that $\lambda^n \geq \delta_x \|B^n u\|$ for all $n \in \mathbb{N}$. Hence $\lambda \geq \delta_x^{1/n} \|B^n u\|^{1/n}$ for all $n \in \mathbb{N}$. This implies that $\lambda \geq \gamma_B(u)$. □

Recall that the orbital spectral radius is defined by $\mathbf{r}_o(B) = \sup_{x\in X_+} \gamma_B(x)$.

Theorem 4.2. *Let X be an ordered normed vector space with normal cone X_+ and $B : X_+ \to X_+$ be homogeneous, order preserving, and bounded, $\mathbf{r}_o(B) > 0$.*

Let $K : X_+ \to X_+$ be continuous and compact and $K(x) \geq Bx$ for all $x \in X$, $\|x\| = 1$.

Then there exists some $x \in X_+$, $\|x\| = 1$, and some $\lambda \geq \mathbf{r}_o(B)$ such that $Kx = \lambda x$. If $B = K$, then $\lambda = \mathbf{r}_o(B)$.

Proof. Since $\gamma_B(\alpha u) = \gamma_B(u)$ for all $u \in X_+$, $\alpha > 0$, we can choose a sequence (u_n) in X_+ such that $\|u_n\| \to 0$ and $\gamma_B(u_n) \to \mathbf{r}_o(B)$. Define

$$K_n(x) = K(x) + u_n, \qquad x \in X_+, n \in \mathbb{N}.$$

By Proposition 4.1, for each $n \in \mathbb{N}$, there exist $x_n \in X_+$ such that $\|x_n\| = 1$ and $\lambda_n \geq \gamma_B(u_n)$ such that

$$\lambda_n x_n = K_n(x_n) = K(x_n) + u_n.$$

Since K is compact, after choosing a subsequence, Kx_n converges to some $y \in X_+$. The corresponding subsequence of (λ_n) is bounded and $\lambda_n \to \lambda \geq \mathbf{r}_o(B) > 0$ after choosing another subsequence. Since $u_n \to 0$, $x_n \to x \in X_+$ with $\|x\| = 1$ and $Kx = \lambda x$. If $B = K$, $\mathbf{r}_o(B) \geq \gamma_B(x) = \lambda$ and so $\mathbf{r}_o(B) = \lambda$. $\square$

Remark 4.3. Let $B = K$. Then $\mathbf{r}_o(B) = \mathbf{r}_+(B)$ by Theorem 1.5 (ii). Further, if X is a Banach space, normality of the cone does not need to be assumed ([28, Thm.2.1] and the remarks in [24]).

5 Monotonically compact maps

Definition 5.1. Let $u \in X_+$, $u \neq 0$, and $B : X_+ \to X_+$ be pointwise u-bounded. B is called *monotonically compact* if (Bx_n) converges for each monotone sequence (x_n) in X_+ for which there is some $c > 0$ such that $x_n \leq cu$ for all $n \in \mathbb{N}$.

If X_+ is regular (Definition 2.7), then every continuous $B : X_+ \to X_+$ is monotonically compact.

Definition 5.2. X_+ is called *minihedral* if $x \wedge y = \inf\{x, y\}$ exists for any $x, y \in X_+$.

Theorem 5.3. *Let X be an ordered normed vector space with a normal minihedral cone X_+. Let $B : X_+ \to X_+$ be continuous, order preserving, and homogeneous. Further let B be monotonically compact and uniformly u-bounded for some $u \in X_+$, $u \neq 0$. Finally assume that $\mathbf{r}_+(B) > 0$.*

Then there exists some $x \in X_+$, $x \neq 0$, such that $Bx \geq \mathbf{r}_+(B)x$ and $\mathbf{r}_o(B) = \mathbf{r}_+(B)$.

The first part of the proof follows [22, L.9.5] almost verbatim where B is assumed to be a linear operator on the ordered Banach space X. It is given here for the ease of the reader and the author's peace of mind that linearity can be safely dropped.

Proof. Let $u \in X_+$ such that B is uniformly u-bounded. Since B is homogeneous, we can assume that $\mathbf{r}_+(B) = 1$ and $u \neq 0$. We define

$$x_0 = u, \qquad x_k = \min\{Bx_{k-1} + 2^{-k}u, u\}, \quad k \in \mathbb{N}.$$

Then $x_k \leq u = x_0$ for all $n \in \mathbb{N}$. By induction, since B is order preserving, $x_{k+1} \leq x_k$ for all $k \in \mathbb{N}$. Since B is monotonically compact, there exists some $z \in X_+$ such that $Bx_k \to z$ as $k \to \infty$ and $Bx_k \geq z$ for all $k \in \mathbb{N}$. Then $y_k := Bx_{k-1} + 2^{-k}u \to z$. Further $y_k \geq Bx_{k-1} \geq z$. Thus,

$$x_k = \min\{y_k, u\} \geq \min\{z, u\} =: x.$$

Notice that

$$\min\{y_k, u\} + z - y_k \leq \min\{z, u\} = x.$$

So $0 \leq x_k - x \leq y_k - z$. Since X_+ is normal and $y_k \to z$, also $x_k \to x$. Since B is continuous, $Bx = z \geq x$. Further $x = \min\{z, u\} = \min\{Bx, u\}$.

It remains to show that $x \neq 0$. Suppose $x_k \to 0$ as $k \to \infty$. Since B is uniformly u-bounded, there exists some $m \in \mathbb{N}$ such that $Bx_{k-1} + 2^{-k}u \leq u$ for all $k \geq m$. Hence

$$x_k = Bx_{k-1} + 2^{-k}u, \qquad k \geq m.$$

In particular, $2^m x_m \geq u$ and $x_k \geq Bx_{k-1}$ for all $k \geq m$. Since B is order preserving and homogeneous,

$$2^m x_{m+n} \geq B^n(2^m x_m) \geq B^n u$$

and

$$2^m Bx_{m+n} \geq B^{n+1}u.$$

Since $x_{m+n} \to 0$ as $n \to \infty$ and B is u-bounded, $2^m B x_{m+n} \leq (1/2)u$ for sufficiently large n. This shows that, for some $n \in \mathbb{N}$, $B^{n+1}u \leq (1/2)u$. By Corollary 3.12, $\mathbf{r}_+(B) < 1$, a contradiction.

This shows that $x \neq 0$ and $Bx \geq x$. By (1.8), $\gamma_B(x) \geq 1$ and $r_o(B) \geq 1 = \mathbf{r}_+(B)$ by (1.9). $\square$

The following definition is similar to the one in [6, III.2.1]

Definition 5.4. An order preserving map $B : X_+ \to X_+$ is called strictly increasing if for any $x, y \in X_+$ with $x \leq y$ and $\|x\| \neq \|y\|$ there exists some $\epsilon > 0$ and some $m \in \mathbb{N}$ such that $B^m(y) \geq (1+\epsilon)B^m x$.

Theorem 5.5. *Let X be an ordered normed vector space with a normal minihedral cone X_+. Let $B : X_+ \to X_+$ be continuous, strictly increasing, and homogeneous. Further let B be monotonically compact and uniformly u-bounded for some $u \in X_+$, $u \neq 0$. Finally assume that $\mathbf{r}_+(B) > 0$.*

Then there exists some $x \in X_+$, $x \neq 0$, such that $Bx = \mathbf{r}_+(B)x$.

Proof. We can assume that $\mathbf{r}_+(B) = 1$. By Theorem 5.3, there exists some $x \in X_+$, $\|x\| = 1$ such that $Bx \geq x$. Then the sequence $(x_n)_{n \in \mathbb{Z}_+}$ in X_+ defined by $x_n = B^n x$ is increasing. We claim that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Suppose that $n \in \mathbb{N}$ is the first n with $\|x_n\| \neq 1$. Then $\|x_{n-1}\| \neq \|x_n\|$. Since B is strictly increasing, there exists some $\epsilon > 0$ and some $m \in \mathbb{N}$ such that $B^m x_n \geq (1+\epsilon)B^m x_{n-1}$. By definition of (x_n), $By \geq (1+\epsilon)y$ for $y = B^m x_{n-1} \geq x$. Since $y \in X_+$ and $y \neq 0$, $\mathbf{r}_+(B) \geq 1+\epsilon$, a contradiction.

Set $x_0 = x$. Then $x_n = Bx_{n-1}$ for $n \in \mathbb{N}$. Since B is uniformly u-bounded, there exists some $c \geq 0$ such that $x_n \leq c\|x_{n-1}\|u = cu$. Since B is monotonically compact, $(Bx_{n-1}) = (x_n)$ converges to some $y \in X_+$, $\|y\| = 1$. Since B is continuous, $By = y$. $\square$

6 Eigenfunctionals

The celebrated Krein-Rutman theorem does not only state the existence of a positive eigenvector but also of a positive eigenfunctional of a positive linear map provided that the map is compact and the cone is total or that the cone is normal and solid. We explore what still can be done if the additivity of the operator is dropped. Recall the left resolvents R_λ, $\lambda > \mathbf{r}_+(B) > 0$, in Section 3.3.

Proposition 6.1. *Let X be an ordered normed vector space with cone X_+ and $B : X_+ \to X_+$ be homogeneous, order preserving, and uniformly order bounded. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$ and that there exists $v \in X_+$ and $x^* \in X_+^*$ such that $x^*(R_\lambda(v)) \to \infty$ as $\lambda \to \mathbf{r}_+$.*

Then there exists a homogeneous, order preserving, bounded nonzero functional $\phi : X_+ \to \mathbb{R}$ such that $\phi \circ B = \mathbf{r}\phi$ and $\phi(u) > 0$ for each order bound u of B.

Our proof will not provide continuity of ϕ.

Proof. Let x^* and v be as above. Choose a sequence (λ_n) in $(\mathbf{r}, \infty)$ such that $x^*(R_{\lambda_n}(v)) \to \infty$ as $n \to \infty$. Define $\psi_n : X_+ \to \mathbb{R}_+$ by

$$\psi_n(x) = x^*(R_{\lambda_n}(x)), \qquad n \in \mathbb{N}, x \in X_+.$$

The functionals ψ_n are homogeneous, order preserving, and continuous, $\psi_n(v) \to \infty$ as $n \to \infty$. By (3.2),

$$\psi_n(Bx) = x^*((R_{\lambda_n}(Bx)) = x^*(R_{\lambda_n}(\lambda_n x) - x) = \lambda_n \psi_n(x) - x^* x.$$

We set

$$\phi_n = \frac{\psi_n}{\|\psi_n\|_+}.$$

Then $\|\phi_n\|_+ = 1$ and

$$\lambda_n \phi_n - \phi_n \circ B = \frac{x^*}{\|\psi_n\|} \to 0, \qquad n \to \infty. \tag{6.1}$$

By Tychonoff's compactness theorem for topological products, there exists some

$$\phi \in \bigcap_{m \in \mathbb{N}} \overline{B_m}, \qquad B_m = \{\phi_n; n \geq m\},$$

where the closure is taken in the topology of pointwise convergence on $\{x \in X_+; \|x\| \leq 1\}$. Notice that all ϕ_n are order preserving, bounded and homogeneous. ϕ inherits these properties. For instance, let $x_1, x_2 \in X_+$ and $x_1 \leq x_2$. Then there exist a strictly increasing sequence (n_j) of natural number such that $\phi_{n_j}(x_i) \to \phi(x_i)$ as $j \to \infty$, $i = 1, 2$. Since $\phi_{n_j}(x_1) \leq \phi_{n_j}(x_2)$ for all $j \in \mathbb{N}$, also $\phi(x_1) \leq \phi(x_2)$.

Similarly, for $x \in X_+$, there exists a strictly increasing sequence (n_j) of natural number such that $\phi_{n_j}(x) \to \phi(x)$ and $\phi_{n_j}(Bx) \to \phi(Bx)$ as $j \to \infty$. By (6.1), $\phi(Bx) = \mathbf{r}\phi(x)$.

We need to rule out that ϕ is the zero functional. Let $u \in X_+$, $\|u\| = 1$, such that B is uniformly u-bounded: There exist some $c \geq 0$ such that $Bx \leq c\|x\|u$ for all $x \in X_+$.

Let $x \in X$, $\|x\| \leq 1$. Since each ϕ_n is order preserving, by (6.1),

$$\lambda_n \phi_n(x) = \phi_n(Bx) + \frac{x^*(x)}{\|\psi_n\|_+} \leq \phi_n(c\|x\|u) + \frac{x^*(x)}{\|\psi_n\|_+} \leq c\phi_n(u) + \frac{\|x^*\|}{\|\psi_n\|_+}.$$

Since this holds for all $x \in X_+$, $\|x\| \leq 1$,

$$\lambda_n = \lambda_n \|\phi_n\|_+ \leq c\phi_n(u) + \frac{\|x^*\|}{\|\psi_n\|_+}.$$

Since $\phi_{n_j}(u) \to \phi(u)$ for some strictly increasing sequence (n_j) in $\mathbb{N}$ and $\|\psi_{n_j}\|_+ \to \infty$,

$$0 < \mathbf{r} \leq c\phi(u).$$ □

Theorem 6.2. *Let X be an ordered normed vector space with cone X_+ and $B : X_+ \to X_+$ be homogeneous, order preserving, and uniformly order bounded. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$ and that there exists some $v \in X_+$, $v \neq 0$, such that $Bv \geq \mathbf{r}v$. Then there exists a homogeneous, order preserving, bounded nonzero functional $\phi : X_+ \to \mathbb{R}$ such that $\phi \circ B = \mathbf{r}\phi$.*

Proof. Let (λ_n) be a sequence in $(\mathbf{r}, \infty)$ such that $\lambda_n \searrow \mathbf{r}$ as $n \to \infty$. Let $v \in X_+$, $\|v\| = 1$ and $Bv \geq \mathbf{r}v$. Choose some $x^* \in X_+^*$ such that $x^*x > 0$ [21, Sec.1.4.1]. Then

$$x^*(R_\lambda(v)) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} x^*(B^k v) \geq \sum_{k=0}^{\infty} \lambda^{-(k+1)} x^*(\mathbf{r}^k v) = \frac{x^* v}{\lambda - \mathbf{r}} \xrightarrow{\lambda \to \mathbf{r}+} \infty.$$

Apply Proposition 6.1. □

The following result is well-known for vectors rather than functionals if B is linear [22, Thm.9.3][29, Thm.2.2].

Lemma 6.3. *Let X be an ordered normed vector space with cone X_+ and $B : X_+ \to X_+$. Let $\mathbf{r} > 0$, $p \in \mathbb{N}$, and $\psi : X_+ \to \mathbb{R}$ with $\psi \circ B^p = \mathbf{r}^p \psi$. Then $\varphi \circ B = \mathbf{r}\varphi$ for*

$$\varphi = \sum_{k=0}^{p-1} \mathbf{r}^{-k} \psi \circ B^k.$$

Proof. For all $x \in X_+$,

$$\varphi(B(x)) = \sum_{k=0}^{p-1} \mathbf{r}^{-k} \psi(B^{k+1}(x)) = \sum_{k=1}^{p-1} \mathbf{r}^{1-k} \psi(B^k(x)) + \mathbf{r}^{1-p} r^p \psi(x) = \mathbf{r}\varphi(x).$$

□

Corollary 6.4. *Let X be an ordered normed vector space with cone X_+ and $B : X_+ \to X_+$ be homogeneous, bounded and order preserving. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$ and that there exist $m, \ell \in \mathbb{N}$ and some $v \in X_+$, $v \neq 0$, such that $B^m v \geq \mathbf{r}^m v$ and B^ℓ is uniformly order bounded. Then there exists a homogeneous, order preserving, bounded nonzero functional $\phi : X_+ \to \mathbb{R}$ such that $\phi \circ B = \mathbf{r}\phi$.*

Proof. Set $p = m\ell$. Then $B^p \geq \mathbf{r}^p v$ and B^p is uniformly order bounded. By Theorem 6.2, there exists some homogeneous, order preserving, bounded nonzero functional $\varphi : X_+ \to \mathbb{R}$ such that $\varphi(B^p(x)) = \mathbf{r}^p \varphi(x)$ for all $x \in X_+$. Apply the previous lemma and notice that φ inherits the desired properties from ψ and B. □

Theorem 6.5. *Let X be an ordered normed vector space with normal cone X_+ and $B : X_+ \to X_+$ be homogeneous, order preserving, continuous, and some power of B is uniformly order bounded, $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then there exists some homogeneous, order preserving, bounded $\phi : X_+ \to R_+$ with $\phi \circ B = \mathbf{r}\phi$ if at least one of the following assumptions is satisfied.*

(a) A power of B is compact.

(b) X_+ is a minihedral cone, and some power of B is monotonically compact.

Proof. (a) Combine Theorem 4.2 with Corollary 6.4.
(b) Combine Theorem 5.5 with Lemma 6.3. □

Theorem 6.6. *Let X be an ordered normed vector space with a normal complete cone X_+. Let B be homogeneous, order-preserving, pointwise order bounded and $B(x + y) \geq B(x) + B(y)$ for all $x, y \in X_+$. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then there exists a homogeneous, concave, order preserving, bounded nonzero functional $\phi : X_+ \to \mathbb{R}$ such that $\phi(B(x)) = \mathbf{r}\phi(x)$ for all $x \in X_+$.*

Proof. By Theorem 3.10, we can assume that some power of B is uniformly order bounded. By Lemma 6.3, we can assume that B itself is uniformly order bounded. By Corollary 3.8, $x^*(R_\lambda(x)) \to \infty$ as $\lambda \to \mathbf{r}+$ with some $x \in X_+$, $x^* \in X_+^*$. Apply Proposition 6.1. $\square$

7 Scent of a beer barrel

We show existence of eigenvectors for two classes of order-preserving homogeneous continuous maps B which are not necessarily compact themselves but have compact powers.

Theorem 7.1. *Let X be an ordered normed vector space with a cone X_+ and $B : X_+ \to X_+$ be homogeneous, order-preserving and continuous. Assume that X_+ is normal or X a Banach space.*

Let Y be a linear subspace of X, $B(X_+) \subseteq Y$, which carries a norm $\|\cdot\|_Y$ such that $\{\|Bx\|_Y;\ x \in X_+, \|x\| \le 1\}$ is bounded and B is compact from $(Y_+, \|\cdot\|_Y)$ into $(X_+, \|\cdot\|)$ where $Y_+ = X_+ \cap Y$.

Assume that $\mathbf{r}_+(B) > 0$ and that there is some $u \in Y_+$, $u \neq 0$, such that B is uniformly u-bounded.

Then there exists some $v \in X_+ \cap Y$, $\|v\| = 1$, such that $Bv = \mathbf{r}_+(B)v$.

The normality of X_+ can be dropped if X is an ordered Banach space.

Lemma 7.2. *Let X be an ordered normed vector space with a cone X_+ and $B : X_+ \to X_+$ be homogeneous, order-preserving and continuous. Assume that X_+ is normal or X a Banach space.*

Let $\psi : X_+ \to \mathbb{R}_+$ that is homogeneous, order preserving, continuous and $\psi(x) > 0$ for all $x \in X_+$, $x \neq 0$. Let $u \in X_+$ and B be uniformly u-bounded.

Set $Q(x) = \psi(x)u$ and $\tilde{B} = B + Q$.

Then the following hold.

(a) $\tilde{B}$ is continuous, homogeneous and order preserving.

(b) For all $x, y \in X_+$ with $x \le y$ and $\psi(x) < \psi(y)$, there exists some $\eta > 0$ such that $\tilde{B}y \ge (1 + \eta)\tilde{B}x$.

Let B satisfy the assumptions of Theorem 7.1. Then

(c) $\tilde{B}^2$ is compact.

(d) There exists a unique $v \in X_+$ such that $\psi(v) = 1$ and $\tilde{B}v = \mathbf{r}_+(B)v$.

Proof. (a) Notice that Q is continuous, homogeneous and order preserving.

(b) We can assume that $x \neq 0$. Since B is uniformly u-bounded, there exists some $c \geq 0$ such that $Bx \leq c\|x\|u$ for all $x \in X_+$. Let $\delta = \frac{\psi(y)-\psi(x)}{2}$. Then

$$\begin{aligned}\tilde{B}y =& Bx + \psi(x)u + 2\delta u \\ \geq& Bx + \psi(x)u + \delta\frac{\psi(x)}{\|\psi\|_+\|x\|}u + \frac{\delta}{(c+1)\|x\|}Bx \geq \eta\tilde{B}(x)\end{aligned}$$

with $\eta > 0$ being the smaller of $\frac{\delta}{\|\psi\|_+\|x\|}$ and $\frac{\delta}{(c+1)\|x\|}$.

(c) Let (x_k) be a bounded sequence in X_+. By assumption (Bx_k) is a bounded sequence in $(Y_+, \|\cdot\|_Y)$. Since $u \in Y \cap X_+$, $(Q(x_k))$ is a bounded sequence in Y and so is $(\tilde{B}x_k)_k$. By the compactness assumption for B, $(B(\tilde{B}x_k))_k$ has a subsequence converging in X_+. Notice that Q is compact from X_+ to Y with the stronger norm. After choosing suitable subsequences, $(B(\tilde{B}x_k))_k$ and $(Q(\tilde{B}x_k))_k$ converge and so does $(\tilde{B}^2x_k)_k$.

(d) Since $\tilde{B}u \geq \psi(u)u$, $\mathbf{r}_o(\tilde{B}) \geq \psi(u) > 0$ by Theorem 3.13. We can assume that $\mathbf{r}_o(\tilde{B}) = 1$. By Theorem 4.2 and Remark 4.3, there exists some $v \in X_+$ such that $\psi(v) = 1$ and $\tilde{B}^2v = v$. Assume that there is some $w \in X_+$ with the same properties.

By construction of $\tilde{B}$,

$$v \geq \psi(Bv + \psi(v)u)u \geq \psi(\psi(v)u)u = \psi(v)\psi(u)u.$$

Since $\psi(u) > 0$ and $\psi(v) > 0$, u is v-bounded. Since B is uniformly u-bounded,

$$v \leq c\|\tilde{B}v\|u,$$

and v is u-bounded. The same holds for w such that v and w are comparable. By construction, $\tilde{B}$ is uniformly u-bounded. So $\tilde{B}$ is uniformly v-bounded and $\mathbf{r}_+(\tilde{B}) = \mathbf{r}_o(\tilde{B})$.

Recall the functional $[\cdot]_v$ from (2.5). Since w is v-comparable, $w \geq [w]_v v$ with $[w]_v > 0$.

Suppose that $\psi(w) > \psi([w]_v v)$. By part (b), there exists some $\eta > 0$ such that

$$\tilde{B}w \geq (1+\eta)\tilde{B}([w]_v v).$$

Then

$$w = \tilde{B}^2w \geq (1+\eta)[w]_v\tilde{B}^2v = (1+\eta)[w]_v v.$$

By definition, $[w]_v \geq (1+\eta)[w]_v$, a contradiction.

This implies $\psi(w) = \psi([w]_v v) = [w]_v \psi(v)$. Since $\psi(v) = 1 = \psi(w)$, $[w]_v = 1$ and $w \geq v$.

By symmetry, $v \geq w$ and $v = w$.

Now set $w = \tilde{B}v$. Since $\tilde{B}w = v \neq 0$, $w \neq 0$. Then $\tilde{B}^2 w = w$. By our previous consideration $\frac{1}{\psi(v)}v = \frac{1}{\psi(Bv)}Bv$. So there exists some $\lambda > 0$ such that $\tilde{B}v = \lambda v$. Then $v = \tilde{B}^2 v = \lambda \tilde{B} v = \lambda^2 v$. This implies $\lambda = 1$ and $\tilde{B}v = v$. $\square$

Proof of Theorem 7.1. Since X_+ is normal, by Proposition 2.3, there is a homogeneous, subadditive, order-preserving functional $\psi : X_+ \to \mathbb{R}_+$ such that $\psi(x) > 0$ for all $x \in X_+$, $x \neq 0$.

Choose a sequence (ϵ_n) of positive numbers such that $\epsilon_n \to 0$. Set $Q_n(x) = \epsilon_n \psi(x) u$ and $B_n = B + Q_n$.

By Lemma 7.2, for each $n \in \mathbb{N}$ there exists a unique $v_n \in X_+$ such that $\psi(v_n) = 1$ and $B_n v_n = \lambda_n v_n$, $\lambda_n = \mathbf{r}_+(B_n)$.

By Theorem 3.3, $\lambda_n \geq \mathbf{r}_+(B)$ and $\lambda_n \leq \mathbf{r}_+(B + \eta\psi(\cdot)u)$ for all $n \in N$, where $\eta = \sup_{n\in\mathbb{N}} \epsilon_n$. After choosing a subsequence we can assume that $\lambda_n \to \lambda \geq \mathbf{r}_+(B) > 0$.

(Bv_n) is a bounded sequence in $(Y_+, \|\cdot\|_Y)$ and so is $(B_n v_n)$ and thus $(\lambda_n v_n)$. Notice that

$$\lambda_n^2 v_n = B(\lambda_n v_n) + \epsilon_n \psi(\lambda_n v_n) u.$$

By the compactness assumption for B, $(B(\lambda_n v_n))$ converges after choosing a subsequence. Since $\epsilon_n \to 0$ and $\lambda_n \to \lambda > 0$, we can assume that (v_n) converges to some $w \in X_+$, $\|w\| = 1$. Since B is continuous, $Bw = \lambda w$ and $\lambda = \mathbf{r}_+(B)$ by Theorem 3.13. $\square$

B is called *essentially compact* if there exists some $k \in \mathbb{N}$ such that $B^k = K + L$ with a continuous compact operators $K : X_+ \to X_+$ and a linear operator bounded operator $L : X \to X$ and $\mathbf{r}(L) < [\mathbf{r}_+(B)]^k$.

Theorem 7.3. *Let X be a normed vector space with a normal and minihedral cone X_+. Let $B : X_+ \to X_+$ be homogeneous, continuous, order preserving, and essentially compact, $\mathbf{r}_+(B) > 0$. Assume that there is some $u \in X_+$, $u \neq 0$, such that B is uniformly u-bounded and monotonically compact. Further assume the following one-sided uniform continuity condition for B:*

For any $\epsilon > 0$ and any $c > 0$ there exists some $\delta > 0$ such that $\|B(x+y) - B(x)\|_u \leq \epsilon$ for all $x, y \in X_+ \cap X_u$ with $\|x\|_u \leq c$ and $\|y\|_u \leq \delta$.

Then there exists some $v \in X_+$, $v \neq 0$, such that $Bv = \mathbf{r}_+(B)v$.

Proof. We can assume that $\mathbf{r}_+(B) = 1$. Since X_+ is normal, there exists an equivalent monotone norm. Let ψ be the restriction of that norm to X_+ and (ϵ_n) be a sequence of positive numbers such that $\epsilon_n \searrow 0$. Define $B_n : X_+ \to X_+$ by $B_n(x) = Bx + \epsilon_n \psi(x)u$. Then B_n is continuous, homogeneous, order preserving, uniformly u-bounded and monotonically compact. B_n also satisfies Lemma 7.2 (b), so B_n is strictly increasing. By Theorem 5.5, there exist eigenvectors $v_n \in X_+$, $\psi(v_n) = 1$, such that

$$\lambda_n v_n = B_n(v_n) = B(v_n) + \epsilon_n u, \qquad \lambda_n = \mathbf{r}_+(B_n). \tag{7.1}$$

By Theorem 3.3, (λ_n) is a decreasing sequence and $\lambda_n \geq 1$. So $\lambda_n \to \lambda$ with $\lambda \geq 1$. Notice that $\lambda_n v_n \geq B v_n$. Thus, for all $k \in \mathbb{N}$, $\lambda_n^k v_n \geq B^k v_n$.

Since ψ is the restriction of an equivalent norm to X_+, the sequence (v_n) is bounded. Since B is uniformly u-bounded, there is some $c \geq 0$ such that

$$v_n \leq \frac{1}{\lambda_n}(B v_n + \epsilon_n u) \leq c\|v_n\| u + \epsilon_1 u, \qquad n \in \mathbb{N}.$$

So (v_n) is a u-bounded sequence in $X_u \cap X_+$. By induction,

$$B_n^k v_n \leq B^k v_n + \alpha_n u, \qquad \alpha_n \to 0. \tag{7.2}$$

This is true for $k = 1$. Assume that $k \in \mathbb{N}$ and it holds for k. Then

$$B_n^{k+1} v_n \leq B(B^k v_n + \alpha_n u) + \epsilon_n \psi(B^k v_n + \alpha_n u) u$$

Now (x_n) with $x_n = B^k v_n$ is a u-bounded sequence in $X_u \cap X_+$. Further, the restriction of B to $X_u \cap X_+$ with u-norm is assumed to be uniformly continuous on every u-bounded subset set of $X_u \cap X_+$.

$$\begin{aligned} B_n^{k+1} v_n \leq & B^{k+1} v_n + B(x_n + \alpha_n u) - B(x_n) + \epsilon_n \psi(B^k v_n + \alpha_n u) u \\ \leq & B^{k+1} v_n + \|B(x_n + \alpha_n u) - B(x_n)\|_u u + \epsilon_n [\psi(B^k v_n) + \alpha_n \psi(u)] u. \end{aligned}$$

So $B_n^{k+1} v_n \leq B^{k+1} v_n + \tilde{\alpha}_n u$ with

$$\tilde{\alpha}_n = \|B(x_n + \alpha_n u) - B(x_n)\|_u + \epsilon_n [\psi(B^k v_n) + \alpha_n \psi(u)].$$

Since B satisfies the uniform u-continuity condition assumed above, $\tilde{\alpha}_n \to 0$.

By (7.1) and (7.2),

$$B^k v_n \le \lambda_n^k v_n \le B^k v_n + \alpha_n u, \qquad \alpha_n \to 0.$$

Since B is essentially compact, there exists some $k \in \mathbb{N}$ such that $B^k = K + L$ with a compact continuous operator $K : X_+ \to X_+$ and a linear bounded operator $L : X \to X$ and $\mathbf{r}(L) < [\mathbf{r}_+(B)]^k = 1$. So

$$Kv_n \le \lambda_n^k v_n - Lv_n \le Kv_n + \alpha_n u, \qquad \alpha_n \to 0.$$

Recall that $\lambda_n \searrow \lambda \ge 1$. We rearrange,

$$Kv_n + (\lambda^k - \lambda_n^k)v_n \le (\lambda^k - L)v_n \le Kv_n + \alpha_n u + (\lambda^k - \lambda_n^k)v_n, \qquad \alpha_n \to 0.$$

Since (v_n) is a bounded sequence, after choosing a subsequence, $Kv_n \to z$, $n \to \infty$, and both the left and the right hand side converge to z. By the squeezing theorem (Corollary 2.6), $(\lambda^k - L)v_n \to z$. Since $\lambda^k \ge 1 > \mathbf{r}(L)$ and $(\lambda^k - L)$ has a continuous inverse, $v_n \to (\lambda^k - L)^{-1} z =: v$.

Since $\lambda_n \to \lambda \ge 1$, by (7.1), $\psi(v) = 1$ and $Bv = \lambda v$. Since $\mathbf{r}_+(B) = 1$, $\lambda = 1$. □

Theorem 7.4. *Let X be a normed vector space with a normal and minihedral cone X_+. Let $B : X_+ \to X_+$ be homogeneous, continuous, order preserving, essentially compact, $\mathbf{r}_+(B) > 0$. Assume that there is some $u \in X_+$, $u \ne 0$, such that B is uniformly u-bounded and monotonically compact. Further assume the following one-sided uniform continuity condition for B:*

> *For any $\epsilon > 0$ and any $c > 0$ there exists some $\delta > 0$ such that $\|B(x + y) - B(x)\| \le \epsilon$ for all $x, y \in X_+$ with $\|x\| \le c$ and $\|y\| \le \delta$.*

Then there exists some $v \in X_+$, $v \ne 0$, such that $Bv = \mathbf{r}_+(B)v$.

Proof. As in the proof of Theorem 7.3, there exist eigenvectors $v_n \in X_+$, $\psi(v_n) = 1$, such that

$$\lambda_n v_n = B_n(v_n) = B(v_n) + \epsilon_n u, \qquad \lambda_n = \mathbf{r}_+(B_n). \tag{7.3}$$

By induction, $B_n^k v_n = B^k v_n + u_n$ with $u_n \in X_+$ and $u_n \to 0$.

Obviously this holds $k = 1$. Assume that $k \in \mathbb{N}$ and the statement holds for k. Then

$$\begin{aligned} B_n^{k+1} v_n =& B(B^k v_n + u_n) + \epsilon_n \psi(B^k v_n + u_n) u \\ =& B^{k+1} v_n + B(x_n + u_n) - B(x_n) + \epsilon_n \psi(x_n + u_n) u \end{aligned}$$

where (x_n) is the bounded sequence $x_n = B^k v_n$. By the uniform continuity condition for B and $u_n \to 0$, $\epsilon_n \to 0$,

$$B_n^{k+1} v_n = B^{k+1} v_n + w_n, \qquad w_n = B(x_n + u_n) - B(x_n) + \epsilon_n \psi(x_n + u_n) u \to 0.$$

The remainder of the proof is the same as for Theorem 7.3. $\square$

8 Application to a spatially distributed two-sex population

The population we consider has individuals of both sexes which form pairs in order to reproduce. Most two-sex population models are formulated in continuous time [18, 19, 20]. Here we consider the case that the mating occurs once a year and that the mating season is short which makes a discrete-time model more appropriate. We also assume that individuals do not live to see two mating seasons.

The spatial habitat of the population is represented by a Borel set $\Omega \subseteq \mathbb{R}^m$. If $f : \Omega \to \mathbb{R}_+$ is an integrable function (with respect to the Lebesgue measure), $f(\xi)$, $\xi \in \Omega$, represents the number of newborns at $\xi \in \Omega$.

The migration operators

In order to take account of the movements of individuals over the year, we consider integral operators K_j, $j = 1, 2$,

$$(K_j f)(\xi) = \int_\Omega k_j(\xi, \eta) f(\eta) d\eta, \qquad f \in \mathcal{M}_+, \xi \in \Omega, j = 1, 2. \tag{8.1}$$

Here $k_j(\xi, \eta) \geq 0$ gives the rate at which individuals that are born at η are female ($j = 1$) or male ($j = 2$) and will be at ξ in the year after. $\mathcal{M}_+$ denotes the set of nonnegative Borel measurable functions on Ω.

In fact, $\int_\Omega k_j(\xi, \eta) d\xi$ is the probability that an individual born at η is female or male, respectively, and will be still alive in the year after.

Assume $k_j : \Omega^2 \to \mathbb{R}_+$ to be Borel measurable and

$$\int_\Omega k_j(\xi,\eta)d\xi \le 1, \qquad j = 1,2, \quad \eta \in \Omega. \tag{8.2}$$

Then the K_j are bounded linear operators on $L^1(\Omega)$ and $\|K_j f\|_1 \le \|f\|_1$ for all $f \in L^1(\Omega)$.

These assumptions are assumed throughout this section without further mentioning.

The mating and birth operator

The mating and birth operator, $F : \mathbb{R}_+^\Omega \times \mathbb{R}_+^\Omega \to R_+^\Omega$, is defined by

$$F(f,g)(\xi) = \phi(\xi, f(\xi), g(\xi)), \qquad f,g \in \mathbb{R}_+^\Omega, \xi \in \Omega.$$

Here $\mathbb{R}_+^\Omega$ is the set of functions on Ω with values in $\mathbb{R}_+$ and $\phi : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is the local mating and birth function. If there are x_1 females and x_2 males at location ξ in Ω, $\phi(\xi,x)$ with $x = (x_1,x_2)$ is the amount of offspring produced at ξ. We equip $\mathbb{R}_+^2$ with the standard order and make the following assumptions which hold throughout the rest of this paper:

- $\phi(\cdot,x)$ is Borel measurable for each $x \in \mathbb{R}_+^2$.
- $\phi(\xi,\cdot)$ is order preserving on $\mathbb{R}_+^2$ for each $\xi \in \Omega$.
- $\phi(\xi,\cdot)$ is homogeneous for each $\xi \in \Omega$,

$$\phi(\xi,\alpha x) = \alpha\phi(\xi,x), \qquad \alpha \ge 0, \xi \in \Omega, x \in \mathbb{R}_+^2.$$

- $\phi(\xi,\cdot)$ is continuous for each $\xi \in \Omega$.
- The function $\psi : \Omega \to \mathbb{R}_+$ defined by $\psi(\xi) = \phi(\xi,1,1)$ is bounded.

One example is given by the harmonic mean

$$\phi(\xi,x) = \beta(\xi)\frac{x_1x_2}{x_1+x_2}, \quad x = (x_1,x_2) \in \mathbb{R}_+ \setminus \{(0,0)\}.$$

Here $\beta : \Omega \to \mathbb{R}_+$ is Borel measurable, and $\beta(\xi)$ is per pair birth rate at ξ. Another example is

$$\phi(\xi,x) = \min\left\{\beta_1(\xi)x_1, \beta_2(\xi)x_2\right\}$$

with two Borel measurable functions $\beta_1, \beta_2 : \Omega \to \mathbb{R}_+$.

Notice that $F : \mathbb{R}_+^\Omega \times \mathbb{R}_+^\Omega \to \mathbb{R}_+^\Omega$ is homogeneous and order-preserving. Here $\mathbb{R}_+^\Omega$ is equipped with with the pointwise order $f \le g$ if $f(\xi) \le g(\xi)$ for all $\xi \in \Omega$. F has only weak positivity and order-preserving properties: If can happen that f, g are not identically zero but $F(f,g)$ is if the supports of f and g have empty intersections. F is the Nemytskii or superposition operator associated with ϕ.

For $x = (x_1, x_2) \in \mathbb{R}_+^2$,

$$\phi(\xi, x_1, x_2) \le \phi(\xi, x_1 + x_2, x_1 + x_2) = \psi(\xi)(x_1 + x_2), \qquad \psi(\xi) := \phi(\xi, 1, 1). \tag{8.3}$$

So

$$F(f,g) \le \psi(f+g), \qquad f, g \in \mathbb{R}_+^\Omega, \qquad \psi(\xi) = \phi(\xi, 1, 1). \tag{8.4}$$

It follows from the boundedness of ψ that F is a continuous map from $L_+^p(\Omega) \to L_+^p(\Omega)$ for every $p \in [1, \infty]$. See [16, Thm.3.4.4].

The next year offspring operator

Our state space of choice is $X_+ = L_+^1(\Omega)$, the cone of $X = L^1(\Omega)$. The next year offspring operator is formally given by

$$B(f) = F(K_1 f, K_2 f), \quad f \in L_+^1(\Omega). \tag{8.5}$$

By (8.3),

$$B(f) \le \psi(K_1 f + K_2 f), \qquad f \in L_+^1(\Omega). \tag{8.6}$$

In order to make the operator B uniformly u-bounded for some $u \in X_+$ we make the following assumption (cf. [21, (2.4)]).

Assumption 8.1. Assume that there exists a function $u \in L^1(\Omega)$ such that

$$\psi(\xi)(k_1(\xi, \eta) + k_2(\xi, \eta)) \le u(\xi)$$

for a.a. $(\xi, \eta) \in \Omega^2$ with respect to the $2n$-dimensional Lebesgue measure.

We will establish that B maps X_+ into $X_u \cap X_+$ where X_u is defined as in Definition 2.10.

Lemma 8.2. *B maps $X_+ = L_+^1(\Omega)$ into $X_u \cap X_+$.*

Further $\{\|Bf\|_u;\ f \in X_+, \|f\|_1 \le 1\}$ is bounded and B is uniformly u-bounded.

Proof. For all $\xi \in \Omega$ and $f \in L^1_+(\Omega)$, by (8.3) and (8.1),

$$\phi(\xi, (K_1 f)(\xi), (K_2 f)(\xi)) \le \psi(\xi) \int_\Omega [k_1(\xi,\eta) + k_2(\xi,\eta)] f(\eta) d\eta$$

with the right hand side being nonnegative and possibly infinite. For all $g \in L^1_+(\Omega)$, by Tonelli's theorem and our assumptions,

$$\begin{aligned}
&\int_\Omega g(\xi)\phi(\xi, (K_1 f)(\xi), (K_2 f)(\xi)) d\xi \\
\le &\int_{\Omega^2} g(\xi)\psi(\xi)[k_1(\xi,\eta) + k_2(\xi,\eta)] f(\eta) d\xi d\eta \\
\le &\int_{\Omega^2} g(\xi) u(\xi) f(\eta) d\xi d\eta = \int_\Omega g(\xi) u(\xi) \|f\|_1 d\xi.
\end{aligned}$$

This shows that $(Bf)(\xi)$ is defined for a.a. $\xi \in \Omega$ and that $Bf \le \|f\|_1 u$ a.e. on Ω. So B maps $X = L^1(\Omega)$ into X_u and $\{\|Bf\|_u; f \in X_+, \|f\|_1 \le 1\}$ is bounded by 1. □

Since B is uniformly u-bounded and monotonically compact (X_+ is regular), we have the following result from Theorem 3.11, Theorem 5.3 and Theorem 6.5.

Theorem 8.3. *Let the Assumptions 8.1 hold. Then*

$$\|B^n u\|^{1/n} \to \mathbf{r}_+(B), \quad n \to \infty.$$

Further there exists some $f \in L^1_+(\Omega)$, $v \ne 0$, such that $B(f) \ge \mathbf{r}_+(B) f$. There also exists some homogeneous order-preserving bounded functional $\phi : X_+ \to \mathbb{R}_+$ such that $\phi(u) > 0$ and $\phi \circ B = \mathbf{r}_+(B)\phi$.

Finally $\mathbf{r}_+(B) = \mathbf{r}_o(B)$.

8.1 An eigenvector with beer barrel scent

Our aim is applying Theorem 7.1 with $Y = X_u$ endowed with the u-norm.

Lemma 8.4. *B is a compact map from $X_+ \cap X_u$ with the u-norm to $X_+ = L^1_+(\Omega)$.*

Proof. Since $u \in L^1(\Omega)$, by Tonelli's theorem,

$$\infty > \int_\Omega \Big(\int_\Omega k(\xi,\eta) d\xi \Big) u(\eta) d\eta = \int_\Omega d\xi \int_\Omega k(\xi,\eta) u(\eta) d\eta.$$

Thus, for a.a. $\xi \in \Omega$, $k_j(\xi,\cdot) \in L^1_+(u d\eta) := L^1(\Omega,\mu)$ where μ is the finite regular measure $\mu(S) = \int_S u(\eta)d\eta$ on the Borel subsets S of Ω. We extend μ to the Borel subsets S of $\mathbb{R}^n$ by $\tilde\mu(S) = \int_S u(\eta)d\eta$. $\tilde\mu$ is then a finite Baire measure on the locally compact space $\mathbb{R}^n$. Since $\mathbb{R}^n$ is σ-compact (countable at infinity) and its topology has a countable base, $C_c(\mathbb{R}^N)$, the space of continuous functions with compact support, is countable under the supremum norm [3, Thm.7.6.3] and dense in $L^1(\mathbb{R}^n, \tilde\mu)$ [3, Cor.7.5.6]. So $L^1(\mathbb{R}^n,\tilde\mu)$ has a dense countable subset. Restriction to Ω provides a dense countable subset $\{g_i; i \in \mathbb{N}\}$ of $L^1(u d\eta)$.

Let (f_ℓ) be a bounded sequence in $X_u \cap X_+$ with the u-norm. Then there exists some $c > 0$ such that $f_\ell \le cu$ for all $\ell \in \mathbb{N}$. For each $i \in \mathbb{N}$, $(\int_\Omega g_i(\eta) f_\ell(\eta) d\eta)_{\ell\in\mathbb{N}}$ is a bounded real sequence which has a convergent subsequence. Using a standard diagonalization procedure, after choosing a subsequence of (f_ℓ), we can assume that $(\int_\Omega g_i(\eta) f_\ell(\eta) d\eta)_{\ell\in\mathbb{N}}$ are convergent sequences for all $i \in \mathbb{N}$. Let $g \in L^1(u d\eta)$ and $\epsilon > 0$. Then there exists some $i \in \mathbb{N}$ such that $\int_\Omega |g(\eta) - g_i(\eta)| u(\eta) d\eta < \epsilon$. For $\ell, m \in \mathbb{N}$,

$$\begin{aligned}
&\Big| \int_\Omega g(\eta) f_\ell(\eta) d\eta - \int_\Omega g(\eta) f_m(\eta) d\eta \Big| \\
\le &\Big| \int_\Omega g(\eta) f_\ell(\eta) d\eta - \int_\Omega g_i(\eta) f_\ell(\eta) d\eta \Big| \\
&\quad + \Big| \int_\Omega g_i(\eta) f_\ell(\eta) d\eta - \int_\Omega g_i(\eta) f_m(\eta) d\eta \Big| \\
+ &\Big| \int_\Omega g_i(\eta) f_m(\eta) d\eta - \int_\Omega g(\eta) f_m(\eta) d\eta \Big| \\
\le &2c \int_\Omega |g(\eta) - g_i(\eta)| u(\eta) d\eta + \Big| \int_\Omega g_i(\eta) f_\ell(\eta) d\eta - \int_\Omega g_i(\eta) f_m(\eta) d\eta \Big|.
\end{aligned}$$

Since $(\int_\Omega g_i(\eta) f_\ell(\eta) d\eta)_\ell$ is a Cauchy sequence,

$$\limsup_{\ell,m\to\infty} \Big| \int_\Omega g(\eta) f_\ell(\eta) d\eta - \int_\Omega g(\eta) f_m(\eta) d\eta \Big| \le 2c \int_\Omega |g(\eta) - g_i(\eta)| u(\eta) d\eta \le 2c\epsilon.$$

Since this holds for each $\epsilon > 0$, $\Big(\int_\Omega g(\eta) f_\ell(\eta) d\eta \Big)_\ell$ is a Cauchy sequence and converges for each $g \in L^1(u d\eta)$. By the consideration of the beginning of

this proof,

$$(K_j(f_\ell)(\xi))_\ell \stackrel{\ell\to\infty}{\longrightarrow} h_j(\xi), \text{ for a.a. } \xi \in \Omega$$

with nonnegative Borel measurable function $h_j : \Omega \to \mathbb{R}_+$, $j = 1, 2$. Since $\phi(\xi, \cdot)$ is continuous for all $\xi \in \Omega$,

$$B(f_\ell)(\xi) \stackrel{\ell\to\infty}{\longrightarrow} F(h_1, h_2)(\xi) \text{ for a.a. } \xi \in \Omega.$$

Since B is homogeneous and order preserving,

$$B(f_\ell) \le cB(u) \in L^1_+(\Omega), \qquad \ell \in \mathbb{N}.$$

Taking the limit as $\ell \to \infty$, $F(h_1, h_2) \le cB(u)$ and

$$|B(f_\ell) - F(h_1, h_2)| \le 2cB(u).$$

By Lebesgue's theorem of a.e. dominated convergence, $\|B(f_\ell)-F(h_1, h_2)\|_1 \to 0$ as $\ell \to \infty$.

This shows that B is compact as a map from $X_u \cap X_+$ with u-norm to X_+. □

Theorem 8.5. *Let Assumption 8.1 be satisfied and* $\mathbf{r}_+(B) > 0$*. Then there exists an eigenfunction* $f \in L^1_+(\Omega)$*,* $f \ne 0$*, such that* $Bf = \mathbf{r}_+(B)f$*.*

Proof. The assertions follow from Theorem 7.1 with $Y = X_u$ whose assumptions have been checked in Lemma 8.2 and Lemma 8.4. In particular, there exists some $f \in X_+$, $\|f\|_1 = 1$, such that $Bf = \mathbf{r}_+(B)f$. □

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